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The Method of Inner Boundary Condition: A New Approach for Solving Singular Perturbation Problems

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The object of this paper is to present a new approach based on the method of inner boundary condition for solving singular perturbation problems. The original problem is partitioned into inner and outer region differential equation systems. Asymptotic expansion is used to obtain the terminal boundary condition. Using an appropriate transformation, a new inner region problem is obtained and solved as a two point boundary value problem. The derivative boundary condition at the terminal point is then derived from the solution of the inner region problem. Using this condition, the outer region problem is efficiently solved by employing the classical finite difference scheme. The proposed method is iterative on the terminal point. Some numerical examples have been solved to demonstrate the efficiency of the method.

1. INTRODUCTION

The main purpose of this paper is to describe a general computational method for solving linear singularly perturbed boundary value problem, P_ε, given by

$$\varepsilon y'' + f(x)\, y' + g(x)\, y = h(x) \tag{1}$$

with appropriate boundary conditions and $0 < \varepsilon \ll 1$. Equations such as this in which the term containing the highest order derivative is multiplied by a small parameter ε, arise frequently in fluid mechanics (boundary layer problems), elasticity (edge effect in shells), quantum mechanics and fluid dynamics, etc. The numerical treatment of singular perturbation problems has always been far from trivial. Pearson [10] was perhaps the first to attempt something like net adjustments in finite difference scheme while treating singularly perturbed equations. Then the idea was developed further by Abrahamsson *et al.* [1] in their study of difference methods to singular perturbation problems. They have shown, in general, that the accuracy cannot be better than $O(\varepsilon)$. Motivated by the asymptotic behavior of singular perturbation problem, Hsiao and Jordan [7] have discussed numerical schemes based on the method of matched asymptotic expansion and modifying the boundary layer problem. Reinhardt [11] also discussed the methods

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based on matched asymptotic expansion. Recently, Roberts [12] has given a boundary value technique for solving singular perturbation problems. There are wide variety of methods based on matched asymptotic expansions to solve singularly perturbed boundary value problems. These can be found in the well known books of Eckhaus [5], O'Malley [9], Van Dyke [13], Nayfeh [8], Cole [3] and Axelsson [2]. Looking at the literature cited above, an interesting but amusing observation that has been made is that some of the workers have attemped to solve the singular perturbation problem in the outer region as a reduced problem obtained by putting $\varepsilon = 0$ and thereby ignoring the contribution due to this term, however small it may be, to the solution of the original problem. Our aim here is to solve the singular perturbation problem, as it is, both in inner as well as outer regions without disturbing the nature of the equation. This method is designed on the basis of the asymptotic behavior of the singular perturbation problem. The original problem is partitioned into inner and outer region differential equation systems. To obtain the terminal boundary condition, asymptotic expansion is used in the outer region with appropriate boundary condition. Using an appropriate transformation, a new inner region problem is obtained and solved as a two point boundary value problem. The derivative boundary condition at the terminal point is then derived from the solution of the inner region problem. Using this condition, the outer region problem is efficiently solved by employing the classical finite difference scheme. Finally, the solutions of inner and outer region problems are combined to obtain an approximate solution to the original problem. The process is to be repeated for various choices of terminal point, until the solution profiles stabilize in both the regions. Some numerical examples have been solved to demonstrate the efficiency of the method. Computational results are compared with the exact solutions.

2. Description of the Method

To be specific, we consider the following singular perturbation problem (SPP):

$$\varepsilon y''(x) + f(x)\, y'(x) + g(x)\, y(x) = h(x) \tag{2}$$

for

$$0 \leqslant x \leqslant 1 \text{ with } y(0) = \alpha \text{ and } y(1) = \beta \tag{3}$$

where ε is a small positive parameter $(0 < \varepsilon \ll 1)$; α, β are given constants; and $f(x)$, $g(x)$, and $h(x)$ are assumed to be sufficiently continuously differentiable functions in $[0, 1]$. Furthermore, we assume that $f(x) \geqslant M > 0$ throughout the interval $[0, 1]$, where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighbourhood of $x = 0$.

As mentioned, we divide the original problem into two problems, an inner region problem and an outer region problem. Let x_p be the terminal point or common point or width or thickness of the inner region (boundary layer). To obtain the terminal boundary condition (i.e., an approximate value of y at the terminal point x_p), we use the aymptotic expansion in the outer region with appropriate boundary condition. As is well known from the singular perturbation theory (see Nayfeh [8]) for the case $f(x)>0$ in [0, 1], the boundary condition at the origin must be dropped and the boundary condition at the other end (i.e., at $x=1$) has to be taken into account in the outer region. Hence, the outer region problem is given by

$$\varepsilon y''(x)+f(x)\, y'(x)+g(x)\, y(x)=h(x) \tag{4}$$

for

$$x_p \leqslant x \leqslant 1 \qquad \text{with} \quad y(1)=\beta. \tag{5}$$

We shall seek an outer solution as an asymptotic expansion in the form

$$y(x)=\sum_{n=0}^{\infty} a_n(x)\, \varepsilon^n \tag{6}$$

where $a_n(x)$ are unknown functions to be determined. Substituting the equation (6) in the equations (4) and (5) we get

$$\begin{aligned}\varepsilon[a_0''+a_1''\varepsilon+a_2''\varepsilon^2+\cdots]+f(x)[a_0'+a_1'\varepsilon+a_2'\varepsilon^2+\cdots]\\+g(x)[a_0+a_1\varepsilon+a_2\varepsilon^2+\cdots]=h(x)\end{aligned} \tag{7}$$

for $x_p \leqslant x \leqslant 1$ with

$$a_0(1)+a_1(1)\,\varepsilon+a_2(1)\,\varepsilon^2\cdots=\beta. \tag{8}$$

Equating the coefficients of various powers of ε in equations (7) and (8), we get

$$f(x)\, a_0'+g(x)\, a_0=h(x) \qquad \text{with} \quad a_0(1)=\beta \tag{9}$$

$$f(x)\, a_n'+g(x)\, a_n=-a_{n-1}'' \qquad \text{with} \quad a_n(1)=0, \text{ where } n=1, 2, 3, \ldots. \tag{10}$$

The solution of the equation (9), if we take account of the boundary condition, is

$$a_0(x)=\left[\exp\left(-\int_1^x \frac{g(\xi)}{f(\xi)}\, d\xi\right)\right]\left[\int_1^x \frac{h(s)}{f(s)} \exp\left(\int_1^s \frac{g(\xi)}{f(\xi)}\, d\xi\right) ds+\beta\right]. \tag{11}$$

Recursively, the functions $a_1(x)$, $a_2(x)$,... can be obtained by solving the equation (10) for $n=1, 2, 3, \ldots$. Thus, the expansion for $y(x)$ given in the equation (6) is obtained. Hence, the terminal boundary condition can be obtained from (6) and denote

$$y(x_p)=\sum_{n=0}^{\infty} a_n(x_p)\, \varepsilon^n=\bar{\alpha}. \tag{12}$$

Since the terminal point x_p is common to the both inner and outer regions, it leads to the inner region problem as a two point boundary value problem:

$$\varepsilon y''(x) + f(x)\, y'(x) + g(x)\, y(x) = h(x) \tag{13.1}$$

for

$$0 \leqslant x \leqslant x_p \qquad \text{with} \quad y(0) = \alpha \text{ and } y(x_p) = \bar{\alpha} \tag{13.2}$$

We choose the transformation

$$x = t\varepsilon \tag{14.1}$$

to create a new inner region problem. By rescaling the equation (13.1) with

$$y(x) = y(t\varepsilon) = Y(t) \tag{14.2}$$

$$y'(x) = \frac{y'(t\varepsilon)}{\varepsilon} = \frac{Y'(t)}{\varepsilon} \tag{14.3}$$

$$y''(x) = \frac{y''(t\varepsilon)}{\varepsilon^2} = \frac{Y''(t)}{\varepsilon^2} \tag{14.4}$$

$$f(x) = f(t\varepsilon) = F(t) \tag{14.5}$$

$$g(x) = g(t\varepsilon) = G(t) \tag{14.6}$$

$$h(x) = h(t\varepsilon) = H(t) \tag{14.7}$$

we obtain the new differential equation for the inner region solutions as,

$$Y''(t) + F(t)\, Y'(t) + \varepsilon G(t)\, Y(t) = \varepsilon H(t). \tag{15}$$

Boundary conditions for the equation (15) are determined by (14.2), (14.1) and (13.2) as,

$$Y(0) = \alpha \qquad \text{and} \qquad Y(t_p) = \bar{\alpha}. \tag{16}$$

We solve this new inner region problem (15) with (16) to obtain the solution over the interval $0 \leqslant t \leqslant t_p$. From this solution, we determine the value of $Y'(t_p)$ and inturn $y'(x_p)$ by using the equation (14.3) and denote it as

$$y'(x_p) = \frac{Y'(t_p)}{\varepsilon} = \bar{\beta}. \tag{17}$$

Returning back to the outer region, we have the outer region problem as a two point boundary value problem:

$$\varepsilon y''(x) + f(x)\, y'(x) + g(x)\, y(x) = h(x) \tag{18}$$

for $x_p \leqslant x \leqslant 1$ with

$$y'(x_p) = \bar{\beta} \tag{19.1}$$

and

$$y(1) = \beta \tag{19.2}$$

We solve this outer region problem (18) with (19.1)–(19.2) by employing the classical finite difference scheme to obtain the solutions over the interval $x_p \leqslant x \leqslant 1$. In this scheme, as usual, we divide the interval $[x_p, 1]$ into N equal subintervals with step size $h = (1 - x_p)/N$, and replace the differential equation (18) by a set of difference equations using the central difference formulae (cf. Fox [6])

$$y_i' \cong \frac{y_{i+1} - y_{i-1}}{2h} \tag{20.1}$$

$$y_i'' \cong \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}. \tag{20.2}$$

The derivative boundary condition (19.1) is also replaced by the corresponding difference equation. Including the difference equation at the terminal point x_p, we have N linear algebraic equations involving $y(x_p)$, $y(x_1)$,..., $y(x_{N-1})$ as unknowns. This algebraic system is in the tridiagonal form, which can be very easily and very efficiently solved by a direct method (for details, see Conte and De Boor [4]).

After solving the both inner and outer region problems, we combine the solutions of inner and outer region problems to obtain an approximate solution to the original problem (2) with (3) over the interval $0 \leqslant x \leqslant 1$.

Repeat the process for different choices of 'x_p' (terminal point), until the solution profiles do not differ materially from iteration to iteration. For computational point of view, we use an absolute error criteria, namely

$$|Y(t)^{(m+1)} - Y(t)^{(m)}| \leqslant \delta; \qquad 0 \leqslant t \leqslant t_p \tag{21}$$

where

$$Y(t)^{(m)} = m\text{th iteration of inner region solution}$$

and

$$\delta = \text{prescribed tolerance bound.}$$

3. Test Examples and Numerical Results

Example 1. Consider the following homogeneous SPP which has earlier been solved by Reinhardt [11] and Roberts [12].

$$\varepsilon y'' + y' + y = 0; \qquad 0 \leqslant x \leqslant 1$$

with

$$y(0)=1 \qquad \text{and} \qquad y(1)=2.$$

The exact solution is given by

$$y_\varepsilon(x)=\frac{(2-e^{r_2})}{(e^{r_1}-e^{r_2})}\,e^{r_1x}+\frac{(e^{r_1}-2)}{(e^{r_1}-e^{r_2})}\,e^{r_2x}$$

where

$$r_1=\frac{-1+\sqrt{1-4\varepsilon}}{2\varepsilon}$$

and

$$r_2=\frac{-1-\sqrt{1-4\varepsilon}}{2\varepsilon}.$$

In the outer region $x_p \leqslant x \leqslant 1$, the problem becomes

$$\varepsilon y''+y'+y=0; \qquad x_p \leqslant x \leqslant 1$$

with

$$y(1)=2.$$

Assuming the solution in the form

$$y(x)=\sum_{n=0}^{\infty} a_n(x)\,\varepsilon^n$$

we get the first order problems as follows:

$$a_0'+a_0=0 \qquad \text{with} \quad a_0(1)=2$$

$$a_n'+a_n+a_{n-1}''=0 \qquad \text{with} \quad a_n(1)=0.$$

By taking only three terms in the expansion, we get

$$a_0(x)=2e^{1-x}$$

$$a_1(x)=2(1-x)\,e^{1-x}$$

$$a_2(x)=2\left(\frac{x^2}{2}-3x+\frac{5}{2}\right)e^{1-x}$$

and hence

$$y(x)=2e^{1-x}+2\varepsilon(1-x)\,e^{1-x}+2\varepsilon^2\left(\frac{x^2}{2}-3x+\frac{5}{2}\right)e^{1-x}.$$

Evaluate $y(x)$ at $x=x_p$ and denote $y(x_p)=\bar{\alpha}$.

By choosing the transformation $x = t\varepsilon$ and by rescaling we get new differential equation in the inner region

$$Y'' + Y' + \varepsilon Y = 0; \qquad 0 \leqslant t \leqslant t_p$$

with

$$Y(0) = \alpha \text{ and } Y(t_p) = \bar{\alpha}.$$

This boundary value problem has analytical solution

$$Y(t) = \frac{(\bar{\alpha} - e^{p_2 t_p})}{(e^{p_1 t_p} - e^{p_2 t_p})} e^{p_1 t} + \frac{(e^{p_1 t_p} - \bar{\alpha})}{(e^{p_1 t_p} - e^{p_2 t_p})} e^{p_2 t}$$

where

$$p_1 = \frac{-1 + \sqrt{1 - 4\varepsilon}}{2}$$

and

$$p_2 = \frac{-1 - \sqrt{1 - 4\varepsilon}}{2}.$$

From this $Y(t)$ we can find $Y'(t)$, which will provide us with $y'(x_p)$; we denote

$$y'(x_p) = \frac{Y'(t_p)}{\varepsilon} = \bar{\beta}.$$

Now coming to outer region again, we have

$$\varepsilon y'' + y' + y = 0; \qquad x_p \leqslant x \leqslant 1$$

with

$$y'(x_p) = \bar{\beta} \qquad \text{and} \qquad y(1) = \beta.$$

This two point boundary value problem is solved using finite difference scheme and the solutions for different values of ε are presented in Tables I and II.

EXAMPLE 2. Consider the following non-homogenous SPP which arises frequently in fluid dynamics. This has earlier been solved by Reinhardt [11]:

$$\varepsilon y'' + y' = 1 + 2x, \qquad 0 \leqslant x \leqslant 1$$

with

$$y(0) = 0 \qquad \text{and} \qquad y(1) = 1.$$

The exact solution is given by

$$y_\varepsilon(x) = \frac{(2\varepsilon - 1)(1 - \exp(-x/\varepsilon))}{(1 - \exp(-1/\varepsilon))} + x(x + 1 - 2\varepsilon)$$

TABLE I

Numerical Results for Example 1, $\varepsilon = 10^{-3}$

$t_p \rightarrow$ x	1 $y(x)$	10 $y(x)$	20 $y(x)$	30 $y(x)$	Exact Solution
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
$2.5(10^{-4})$	2.5528358	1.9803880	1.9803425	1.9803423	1.9803425
$5.0(10^{-4})$	3.7620445	2.7438053	2.7437245	2.7437243	2.7437244
$1.0(10^{-3})$	5.4365691	3.8009368	3.8008070	3.8008066	3.8008070
$5.0(10^{-3})$		5.3849677	5.3847644	5.3847638	5.3847643
$1.0(10^{-2})$		5.3878108	5.3876073	5.3876067	5.3876072
$2.0(10^{-2})$			5.3341478	5.3341475	5.3341478
$3.0(10^{-2})$				5.2810192	5.2810193
$4.0(10^{-2})$					5.2284199
$1.0(10^{-1})$	4.9236615	4.9235989	4.9235755	4.9235589	4.9236444
$2.0(10^{-1})$	4.4546650	4.4545913	4.4545907	4.4545904	4.4546513
$3.0(10^{-1})$	4.0303420	4.0302835	4.0302835	4.0302835	4.0303313
$4.0(10^{-1})$	3.6464375	3.6463921	3.6463921	3.6463921	3.6464292
$5.0(10^{-1})$	3.2991012	3.2990671	3.2990671	3.2990671	3.2990950
$6.0(10^{-1})$	2.9848500	2.9848253	2.9849253	2.9848253	2.9848455
$7.0(10^{-1})$	2.7005325	2.7005156	2.7005156	2.7005156	2.7005293
$8.0(10^{-1})$	2.4432971	2.4432869	2.4432869	2.4432869	2.4432951
$9.0(10^{-1})$	2.2105642	2.2105596	2.2105596	2.2105596	2.2105633
1.0	2.0000000	2.0000000	2.0000000	2.0000000	2.0000000

TABLE II

Numerical Results for Example 1, $\varepsilon = 10^{-4}$

$t_p \rightarrow$ x	1 $y(x)$	10 $y(x)$	20 $y(x)$	30 $y(x)$	Exact Solution
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
$2.5(10^{-5})$	2.5525396	1.9813069	1.9812623	1.9812623	1.9812624
$5.0(10^{-5})$	3.7616268	2.7455385	2.7454593	2.7454593	2.7454593
$1.0(10^{-4})$	5.4365637	3.8042071	3.8040798	3.8040798	3.8040798
$1.0(10^{-3})$		5.4316725	5.4314709	5.4314709	5.4314709
$2.0(10^{-3})$			5.4262430	5.4262430	5.5262430
$3.0(10^{-3})$				5.4208189	5.4208189
$4.0(10^{-3})$					5.4154003
$1.0(10^{-1})$	4.9190618	4.9196396	4.9196401	4.9196396	4.9196491
$2.0(10^{-1})$	4.4509658	4.4514307	4.4514308	4.4514307	4.4514380
$3.0(10^{-1})$	4.0274189	4.0277814	4.0277818	4.0277819	4.0277874
$4.0(10^{-1})$	3.6441669	3.6444516	3.6444520	3.6444516	3.6444563
$5.0(10^{-1})$	3.2973860	3.2976039	3.2976043	3.2976039	3.2976075
$6.0(10^{-1})$	2.9836106	2.9837662	2.9837665	2.9837662	2.9837688
$7.0(10^{-1})$	2.6996907	2.6997969	2.6997971	2.6997969	2.6997986
$8.0(10^{-1})$	2.4427893	2.4428533	2.4428535	2.4428533	2.4428544
$9.0(10^{-1})$	2.2103645	2.2103634	2.2103636	2.2103634	2.2103639
1.0	2.0000000	2.0000000	2.0000000	2.0000000	2.0000000

In the outer region, the problem becomes

$$\varepsilon y'' + y' = 1 + 2x; \qquad x_p \leqslant x \leqslant 1$$

with

$$y(1) = 1.$$

Assuming the solution $y(x)$ in the form

$$y(x) = \sum_{n=0}^{\infty} a_n(x)\, \varepsilon^n$$

we get the first order equations

$$a_0' = 1 + 2x \qquad \text{with} \quad a_0(1) = 1$$

$$a_n' = -a_{n-1}'' \qquad \text{with} \quad a_n(1) = 0, \text{ where } n = 1, 2, \ldots.$$

Taking only two terms in the expansion of $y(x)$ we get

$$a_0(x) = x + x^2 - 1$$

$$a_1(x) = 2(1 - x)$$

and hence,

$$y(x) = x + x^2 - 1 + 2\varepsilon(1 - x).$$

Evaluate $y(x)$ at $x = x_p$ and denote

$$y(x_p) = \bar{\alpha}.$$

By choosing the transformation $x = t\varepsilon$ and by rescaling, we get new differential equation in the inner region,

$$Y'' + Y' = \varepsilon + 2\varepsilon^2 t, \qquad 0 \leqslant t \leqslant t_p$$

with

$$Y(0) = 0 \qquad \text{and} \qquad Y(t_p) = \bar{\alpha}.$$

This two point boundary value problem also has analytical solution

$$Y(t) = A + Be^{-t} + \varepsilon t + \varepsilon^2 t^2 - \varepsilon - 2\varepsilon^2 t + 2\varepsilon^2,$$

where the constants A and B are given by

$$B = \frac{(\bar{\alpha} - \varepsilon t_p - \varepsilon^2 t_p^2 + 2\varepsilon^2 t_p)}{(e^{-t_p} - 1)}$$

TABLE III

Numerical Results for Example 2, $\varepsilon = 10^{-3}$

$t_p \rightarrow$ x	1 $y(x)$	10 $y(x)$	20 $y(x)$	30 $y(x)$	Exact Solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
$2.5(10^{-4})$	−0.34898259	−0.22051727	−0.22050725	−0.22050725	−0.22050726
$5.0(10^{-4})$	−0.62071514	−0.39220097	−0.39218314	−0.39218314	−0.39218315
$1.0(10^{-3})$	−0.99700100	−0.62988596	−0.62985732	−0.62985732	−0.62985732
$5.0(10^{-3})$		−0.98630553	−0.98626053	−0.98626053	−0.98626053
$1.0(10^{-2})$		−0.98792000	−0.98787469	−0.98787469	−0.98787469
$2.0(10^{-2})$			−0.97764000	−0.97764000	−0.97764000
$3.0(10^{-2})$				−0.96716000	−0.96716000
$4.0(10^{-2})$					−0.95648000
$1.0(10^{-1})$	−0.88820002	−0.88820860	−0.88819764	−0.88820346	−0.88820001
$2.0(10^{-1})$	−0.75840002	−0.75840015	−0.75839998	−0.75840005	−0.75840001
$3.0(10^{-1})$	−0.60860002	−0.60860004	−0.60860004	−0.60860001	−0.60860001
$4.0(10^{-1})$	−0.43880005	−0.43880006	−0.43880007	−0.43880006	−0.43880001
$5.0(10^{-1})$	−0.24900011	−0.24900008	−0.24900008	−0.24900008	−0.24900001
$6.0(10^{-1})$	−0.03920017	−0.03920008	−0.03920008	−0.03920008	−0.03920000
$7.0(10^{-1})$	0.19059982	0.19059993	0.19059993	0.19059993	0.19060000
$8.0(10^{-1})$	0.44039984	0.44039994	0.44039994	0.44039994	0.44039998
$9.0(10^{-1})$	0.71019990	0.71019997	0.71019997	0.71019997	0.71019999
1.0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000

and

$$A = -B + \varepsilon - 2\varepsilon^2.$$

From this $Y(t)$ we can find $Y'(t)$, which will provide us with $Y'(x_p)$; we denote

$$y'(x_p) = \frac{Y'(t_p)}{\varepsilon} = \bar{\beta}.$$

Now coming to outer region again, we have

$$\varepsilon y'' + y' = 1 + 2x, \qquad x_p \leqslant x \leqslant 1$$

with

$$y'(x_p) = \bar{\beta} \qquad \text{and} \qquad y(1) = 1.$$

This two point boundary value problem is solved using finite difference scheme and the solution for different values of ε are presented in the Tables III and IV.

TABLE IV

Numerical Results for Example 2, $\varepsilon = 10^{-4}$

$t_p \rightarrow$ x	1 $y(x)$	10 $y(x)$	20 $y(x)$	30 $y(x)$	Exact Solution
0.0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
$2.5(10^{-5})$	−0.34983702	−0.22114003	−0.22112999	−0.22112999	−0.22112999
$5.0(10^{-5})$	−0.62228481	−0.39335851	−0.39334065	−0.39334065	−0.39334064
$1.0(10^{-4})$	−0.99970001	−0.63192284	−0.63189416	−0.63189416	−0.63189415
$1.0(10^{-3})$		−0.99879920	−0.99875381	−0.99875381	−0.99875381
$2.0(10^{-3})$			−0.99779640	−0.99779639	−0.99779639
$3.0(10^{-3})$				−0.99679160	−0.99679159
$4.0(10^{-3})$					−0.99579480
$1.0(10^{-1})$	−0.88982894	−0.88982006	−0.88982006	−0.88982006	−0.88982000
$2.0(10^{-1})$	−0.75984900	−0.75984006	−0.75984006	−0.75984006	−0.75984000
$3.0(10^{-1})$	−0.60986901	−0.60986005	−0.60986005	−0.60986005	−0.60986000
$4.0(10^{-1})$	−0.43988874	−0.43988006	−0.43988006	−0.43988006	−0.43988000
$5.0(10^{-1})$	−0.24990813	−0.24990012	−0.24990012	−0.24990012	−0.24990000
$6.0(10^{-1})$	−0.03992718	−0.03992017	−0.03992017	−0.03992017	−0.03991999
$7.0(10^{-1})$	0.19005410	0.19005981	0.19005981	0.19005981	0.19006000
$8.0(10^{-1})$	0.44003573	0.44003983	0.44003983	0.44003983	0.44003999
$9.0(10^{-1})$	0.71001770	0.71001990	0.71001990	0.71001990	0.71002001
1.0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000

4. Discussion

A new method based on the method of inner boundary condition for solving singular perturbation problems is presented. In this method, the solution of the outer region problem provides the terminal condition $(y(x_p))$ for the inner region problem. And in turn, the solution of the inner region problem provides the terminal condition $(y'(x_p))$ for the outer region problem. This serves as the link between the two regions. As mentioned, the method is iterative on the terminal point. The process is to be repeated for various choices of x_p (terminal point) until the solution profiles stabilize in both the regions. The point x_p is not unique but can assume a wide range of values. To reduce the amount of computation, we choose the smallest value of x_p which produces the required accuracy. Because the inner region problem interval is very small relative to the entire interval of the original problem, we can usually improve our accuracy by making x_p larger. As an alternative to the solution of the outer region problem (18) with (19.1)–(19.2), we may use the solution (6) of the problem (4) with (5) over the interval $x_p \leqslant x \leqslant 1$. Two test examples have been solved. We have tabulated the numerical results obtained by the present method as well as the exact solution for different values of ε. The numerical experimentation on these examples demonstrates that the present method approximates the exact solution well. All the calculations have been performed on DEC–10 computer system at IIT Kanpur.

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